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Semiclassical quantization of a separatrix map

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Semiclassical quantization of a separatrix map is presented. An effective Hamiltonian is constructed as a function of the action-angle canonical pair. The quantum dynamics of the system is analyzed for the case of high-frequency perturbation. An explicit form of the Floquet evolution operator is obtained in the unperturbed basis. Quasienergy level spacing statistics is studied. It is shown that the statistics is Poissonian as a result of the bounding nature of the Floquet matrix. This effect indicates a quantum localization process as well. [S1063-651X(96)50607-0]

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Investigation of chaotic motion inside a separatrix layer has a long history but is still of interest in classical and quantum dynamics. The chaotic phenomenon produced by a separatrix map is used to describe a variety of problems such as limited and unlimited diffusion, both classical and quantum, in a separatrix mesh [1,2], current-driven Josephson junction [3], two-dimensional (2D) electronic motion in superlattices [4], breakdown of adiabatic invariance due to quantum dynamics near a classical separatrix in a double-well potential [5]. The separatrix map was obtained by Zaslavsky and co-workers in the investigation of a separatrix splitting [6] in the framework of the Hamiltonian [7]

$$H=\frac{p^2}{2}-\cos x-\varepsilon\ \cos(x-\lambda t)\equiv H_0+\varepsilon V(x,t), \qquad (1)$$

where (p,x) is dimensionless momentum-coordinate canonical pair, t is dimensionless time, while ε is the strength of a perturbation and λ is its frequency. Calculating energy change upon a period T of the unperturbed motion described by H_0 one obtains the Poincaré-Melnikov integral [8]. It reads

$$\Delta E=-\epsilon\int_0^T\frac{\partial H_0}{\partial p}\,\frac{\partial V}{\partial x}dt.$$

Evaluating this integral in the vicinity of a separatrix, one obtains the separatrix map in the following form:

$$\overline{E}=E+\epsilon(\lambda)\sin\lambda t,$$
$$\overline{t}=t+T(\overline{E}), \qquad (2)$$

where $T\equiv T(E)=\ln(32/|E|)$ and $\epsilon(\lambda)\simeq 2\pi\varepsilon\lambda^2e^{-\pi\lambda/2}$ [9]. The overbar for the energy in the second equation in (2) is taken so that the map is Hamiltonian. In Ref. [9] this equation was obtained as well under investigation of separatrix dynamics in the framework of the standard model.

A quantum mechanical counterpart of the system (2) was considered by Bubner and Graham for the case of high-frequency perturbation $\lambda\gg 1$ [3]. The problem of quantum transport inhibition by a localization effect has been considered in the framework of energy-time quantization. It has been stressed in [3] that the main deficiency of such an approach is an appearance of an unphysical time parameter [10,11].

In this paper an approach enabling the use semiclassical quantization in the framework of the action-angle variables for the system (2) is presented. An effective Hamiltonian as a function of the action-angle pair $(I,\theta)$and the same real

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time t is constructed. The effective Hamiltonian describes meaningfully the dynamics of the system (2). The quantum dynamics of the system is analyzed in the framework of classical high-frequency expansion and a semiclassical initial-value representation method [12,13]. It results in an explicit form of the Floquet evolution operator in the unperturbed basis. Level spacing distribution of a quasienergy spectrum is studied. A Poisson distribution of energy levels is found. It indicates a quantum localization process, reported in Ref. [3] as well. In the case of the high-frequency perturbation a local approximation by the standard map is valid [9]. Following that, we obtain

$$\Delta E = \overline{\mathcal{E}} - \mathcal{E} = -\epsilon(\lambda)\sin\lambda t;$$
$$\overline{t} = t + \frac{2\pi k}{\lambda} - \frac{\overline{\mathcal{E}}}{E_r}, \tag{3}$$

where $E = E_r + \mathcal{E}$, $\mathcal{E} \ll E_r$. The resonant energy E_r, determined by the resonance condition $\lambda T(E_r) = 2\pi k$, is

$$E_r = 32\exp\left(-\frac{2\pi k}{\lambda}\right). \tag{4}$$

The map (3) is much simpler for analysis, and still possesses the main features of the original separatrix map. In the absence of the perturbation, $\mathcal{E}$ is energy of a periodic motion with a frequency

$$\omega(I) = \frac{d\mathcal{E}}{dI} = \frac{2\pi}{T(\mathcal{E})} = 2\pi\left(\frac{2\pi k}{\lambda} - \frac{\mathcal{E}}{E_r}\right)^{-1}.$$

Solving this equation, and taking into account the condition $\lambda^2 I/\pi k^2 E_r \ll 1$, we obtain finally an effective Hamiltonian of the unperturbed motion:

$$\mathcal{H}_0 \equiv \mathcal{E} = \Omega I - \frac{\mu I^2}{2}, \tag{5}$$

where

$$\Omega = \frac{\lambda}{k}, \quad \mu = \frac{\lambda^3}{\pi k^3 E_r}. \tag{6}$$

When the separatrix is approached, i.e., in the limit $I \to I_{\max}$, the frequency vanishes $[\omega(I) \to 0]$. The energy change resulting from the perturbation over time $\Delta t = T$ reads from (3) and (5) as

$$\Delta\mathcal{H}_0 = \int_{\Delta t} \omega(I)\dot{I}\,dt = -\epsilon(\lambda)\sin\lambda t$$
$$= -\frac{\epsilon(\lambda)}{\lambda}\int_{\Delta t}\left(\frac{d}{dt'}\delta_{2\pi}(t'-t)\right)\cos\lambda t'\,dt', \tag{7}$$

where $\delta_{2\pi}(z)$ is periodic with a period-2π δ function. Comparing integrands from the left and right hand sides of Eq. (7) we obtain for the arbitrary Δt the following equations of motion for the action-angle pair:

$$\dot{I} = -\frac{\epsilon(\lambda)}{\lambda}\omega(I)\delta'_{2\pi}(\theta)\cos\lambda t,$$
$$\dot{\theta} = \omega(I), \tag{8}$$

where $\delta'_{2\pi}(\theta) \equiv (d/d\theta)\delta_{2\pi}(\theta)$. These equations of motion are not Hamiltonian. To obtain such equations the perturbation of the form $\{[\epsilon(\lambda)]/\lambda\}\omega'(I)\delta_{2\pi}(\theta)\cos\lambda t$ is added to the second equation of (8). Finally, chaotic dynamics near the separatrix is described by the following effective Hamiltonian:

$$\mathcal{H} = \mathcal{H}_0 + \frac{\epsilon(\lambda)}{\lambda}\omega(I)\delta_{2\pi}(\theta)\cos\lambda t. \tag{9}$$

It is simple to show from (8) and (9) that an influence of adding the perturbative term is negligibly small, when $\epsilon(\lambda)\omega'(I)/\omega(I) \ll 1$. It is worthwhile to note in this connection that the procedure described above is quite general for a wide class of the Kepler-like maps.

For the following analysis we replace $\omega(I)$ by Ω in the Hamiltonian (9). For long-term dynamics all these errors accumulate. For the short time scale of the order of the period of the perturbation, upon which the Floquet evolution operator is obtained below, this error does not affect the dynamics, but simplifies the following quantum analysis. It should be stressed that the effective Hamiltonian (9) describes the resonances taking place in (3) with the same criterion for extended chaos. It follows from (3) and (9) that $\lambda T(E_r) = 2\pi k$ and $l\omega(I_r) = \lambda$, where k and l are integers, while I_r is a resonant action corresponding to the lth resonance. The simplification carried out above slightly changes the criterion: $K = (2/\pi)K_0$, where $K_0 = [\lambda\epsilon(\lambda)]/E_r$ is the critical value found by the Chirikov criterion for chaos in the standard map (3).

The Hamiltonian formulation of the problem allows semiclassical quantization in the action-angle formulation: $I \to \hat{I} = \hbar\hat{n} = -i\hbar(\partial/\partial\theta)$. Here $\hbar$ is a dimensionless Planck constant defined from the number of states inside the energy shell. Semiclassical consideration requires that the width of the perturbative potential is larger than the de Broglie wavelength. In this connection it is necessary to restrict summation in the Fourier expansion of the $\delta_{2\pi}(\theta)$ potential. Then we obtain a new potential with the width of a spike equal to $2\pi/N$: $\delta_N(\theta) = 2\sum_{k=0}^{N}\cos k\theta - 1$, where $\delta_N(\theta)$ tends to $\delta_{2\pi}(\theta)$ as N tends to infinity. It is simple to show from the following analysis that

$$f(0) = \int_a^{2\pi+a} f(\theta)\delta_{2\pi}(\theta) = \int_a^{2\pi+a} f(\theta)\delta_N(\theta) + O(1/N),$$

where $f(x)$ is a periodic function with a period 2π, and $1/\hbar \gg N \gg 1$. Terms of order $O(1/N)$ are omitted in the following analysis.

The eigenvalue equation for the Floquet evolution operator $\hat{U} = \hat{T}\exp\{-(i/\hbar)\int_0^{\tau}\mathcal{H}(t')dt'\}$ in the n representation reads

$$\sum_{n'} U_{n,n'}\psi_{n'} = e^{-i\chi_n}\psi_n, \tag{10}$$

where ψ_n is a quasienergy function with a corresponding quasienergy χ_n, $\hat{T}$ is the time ordering operator, and $\tau = 2\pi/\lambda$ is a period of the perturbative field. The initial-value representation [12,13] for the transition amplitude $K(\hbar n';\hbar n) = (1/\hbar)U_{n,n'}$ is used for the following semiclassical analysis [14]. In this case the semiclassical propagator takes the form [12]

$$K(I',t';I,t)\approx\frac{1}{2\pi\hbar}\int_0^{2\pi}\left(\frac{\partial\theta^f(\theta_0)}{\partial\theta_0}\right)^{1/2}\exp\left\{\frac{i}{\hbar}F(\theta_0)\right\}d\theta_0, \tag{11}$$

where

$$F(\theta_0)=[I^f(\theta_0)-I']\theta^f(\theta_0)-\int_t^{t'}d\tau'\{\theta(\tau')\dot{I}(\tau')+\mathcal{H}[I(\tau'),\theta(\tau')]\}, \tag{12}$$

and the condition $\partial\theta^f(\theta_0)/\partial\theta_0\neq 0$ is assumed. Indices 0 and f denote the initial and the final points, respectively, for a classical trajectory determined by the Hamiltonian (9). Solving the classical equations for the period τ one obtains in the leading order of the perturbation theory

$$I^f\equiv I(\tau)\approx\begin{cases} I_0, & \tau<t^* \\ I_0+\dfrac{\epsilon(\epsilon\lambda)}{\Omega}\sin\lambda t^*, & \tau\geqslant t^*,\end{cases} \tag{13}$$
$$\theta^f\equiv\theta(\tau)\approx\theta_0+\omega(I_0)\tau.$$

Here $t^*(\theta_0)=(2\pi-\theta_0)/\omega$ determines a time interval of the free motion before the kick. Substituting (13) in (11) and (12) one writes the matrix elements of the Floquet operator in the form

$$U_{n,n'}=\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{\frac{i}{\hbar}F(\theta_0)\right\}d\theta_0,$$

$$\frac{1}{\hbar}F(\theta_0)=-\frac{2\pi}{\hbar\lambda}\mathcal{H}_1(\hbar n)+\frac{\epsilon}{\hbar\lambda}\cos\lambda t^*(\theta_0)+(n-n')\theta_0+\frac{2\pi\Omega}{\lambda}(n-n'), \tag{14}$$

where the condition $\partial\theta^f/\partial\theta_0=1$ following from (13) is taken into account. Carrying out the integral in (14) in the stationary phase approximation one obtains

$$U_{n,n'}=e^{-i[(2\pi\mathcal{H}_1)/\hbar\lambda]}\times\begin{cases} 1+\dfrac{\Omega}{\lambda}[J_0(\kappa)-1] & \text{if } n=n' \\ \dfrac{\Omega}{\lambda}e^{i\pi\Omega(n-n')/2\lambda}J_\nu(\kappa) & \text{if } n>n' \\ \dfrac{\Omega}{\lambda}e^{i3\pi\Omega(n-n')/2\lambda}J_{|\nu|}(\kappa) & \text{if } n<n',\end{cases} \tag{15}$$

where $\nu=\omega(n-n')/\lambda$ and $\kappa=\epsilon/\hbar\lambda$. Matrix elements are exponentially small for $|\nu|\geqslant\kappa$ and $\kappa\gg1$. It follows from this that the maximal number of nonvanishing off-diagonal elements in a string or column is determined by the expression $\Delta n\equiv|n-n'|=\epsilon/\hbar\Omega=\epsilon k/\hbar\lambda$. Using the definition of the number states in the energy shell, $\mathcal{N}=\hbar^{-1}\int_0^{E_r}T(\mathcal{E})d\mathcal{E}\sim I_{\max}/\hbar$, we can define the condition for the level clustering. It reads as $\zeta=\Delta_n/\mathcal{N}<1$. Taking into account that $K=\epsilon\lambda/E_r$ and $K\sim\lambda^2/4$, one obtains

$$\zeta=\frac{4K}{\pi\lambda^2}. \tag{16}$$

In the case when $\lambda>\pi$, it follows from (16) that $\zeta<1$ and, hence, the matrix U is bounded. Therefore, quasienergies χ have a Poisson distribution when $K>1$ as well [15–18]. Numerical diagonalization of the matrix $U(375\times375)$ and statistical analysis χ show the Poisson level spacing distribution of quasienergies for $\pi\leqslant\lambda\leqslant\frac{4}{3}\pi$ and $1.3<K<3\lambda$.

Condition (16) shows that the quantum dynamics inside the separatrix energy shell does not obey to the conditions of "maximal quantum chaos" [18], namely, the localization length in the energy space exceeds the energy scale. Hence the clustering of the quasienergies takes place, the quasienergies are not ergodic, and, therefore, their statistics is Poissonian even for strong chaotic motion of the classical counterpart. Of course, we can consider the conditions when K is large enough that $\zeta\gg1$, and the condition for "maximal quantum chaos" takes place. But in this case the number of levels inside the stochastic layer is so small (or it is empty) that it is meaningless to speak about any statistics in the quasienergy spectrum. Mathematically, at fixed λ and ϵ, the stochasticity parameter K tends to infinity when E_r tends to zero. Hence the number of states inside the separatrix energy shell $N\propto E_r$ tends to zero.

Finally the following limitation should be noted. When the number of levels inside the energy shell is small, a semiclassical approach is not valid. This means that in the case when $K\gg1$ the system cannot be quantized semiclassically in the framework of the action-angle formulation $[I\to -i\hbar(\partial/\partial\theta)]$. This conclusion is based on the fact that the canonical transformation

$$\left(-i\frac{\partial}{\partial x},x\right)\to\left(-i\frac{\partial}{\partial\theta},\theta\right)$$

analogous to the classical one $(p,x)\to(I,\theta)$ can be made only in the semiclassical limit [19]. So, this quantum problem for $K\gg1$ cannot be studied in the framework of the local approximation (3) but only in the framework of the initial separatrix map problem (2).

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